gmp
M.U.
M.Sc. 94

Cauchy's Fundamental theorem:-

Q.No→ State and Prove Cauchy's theorem.

Ans.→ Statement:- If $f(z)$ is a analytic function of z and if $f'(z)$ is continuous at each point within and on a closed contour 'C', then

$$\int_C f(z)\,dz = 0.$$

Proof:- Let D be the region which consists of all points within and on the contour C. If,

$$P(x, y),\ Q(x, y) = \frac{\partial Q}{\partial x},\ \frac{\partial P}{\partial y}$$

are all continuous functions of x & y in the region D, then Green's theorem states that,

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{[by Green's theorem]}$$

Since $f(z) = u + iv$ is continuous on the simple curve C and $f'(z)$ exists and is continuous in D, therefore u, v, u_x, v_x, v_y are all continuous in D. The condition of Green's theorem are thus satisfied

$$\therefore \int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

(by virtue of the above result of ~~Gr~~ Green's theorem).

$\int_C f(z)dz = 0$ [by Cauchy - Riemann equations]

$$\therefore \int_C f(z)\,dz = 0.$$

Cauchy's Theorem (Revised form) or Cauchy Goursat theorem):

QN.→ State and Prove Cauchy's Goursat theorem.

Ans.→ Statement:- If a function $f(z)$ is analytic and one valued inside and on a simple closed contour C, then,

$$\int_C f(z)\,dz = 0.$$

Proof of lemma:- Before given the ~~Prov~~ ~~Proof~~ Proof ~~the main~~ the main theorem, we shall prove that following theorem which state that statement of lemma, Let $f(z)$ be analytic within a closed contour C. Then for every $\epsilon > 0$, it is always possible to divide the region within C into a finite number n of squares and partial squares whose boundaries are denoted by S_i $(i = 1, 2, ---- n)$ such that there exists a point z_i within each S_i such that

$$\left|\frac{f(z)-f(z_i)}{z-z_i}-f'(z_i)\right|<\epsilon$$

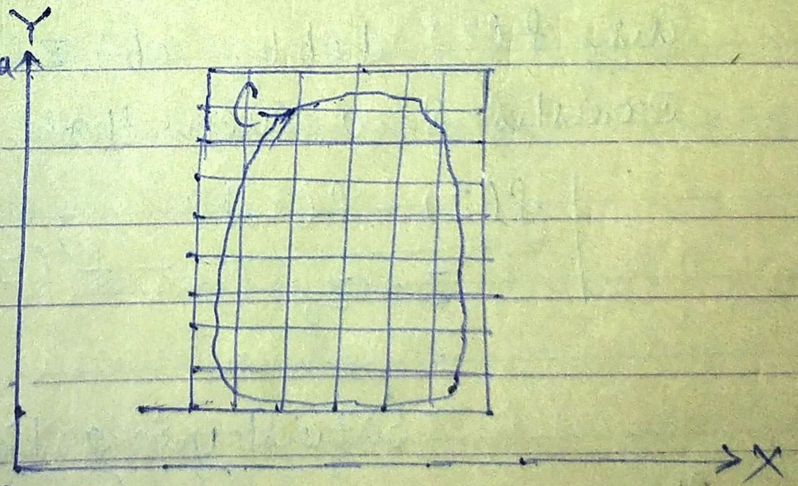
bor each Point $z \neq z_i$ Within or on S_i $(i=1,2,---n)$,

Proof of the lemma: - If Possible let lemma is false $\exists\ \epsilon>0$, Such that however subdivide the region Within C. there Will be at least one mesh (either Square or Partial Square) bor Which

$$\therefore \left|\frac{f(z)-f(z_i)}{z-z_i}-f'(z_i)\right|<\epsilon \quad \text{—— (1)}$$

is not true. We denote by D the closed region consisting of Points Within on C. Let the region D, be Covered by a network of finite number of meshes (Squares or Partial Squares)

Since the ~~lemma~~ lemma is false there will be at ~~least~~ least one mesh (either Square or Partial Square) bor which ~~one~~ (1) is false. We denoted this mesh by C_0.



If this mesh the Square then be take is C_0 and this mesh is Partial Square C_0 denote entire Square Containing it. We divide C_0 into four equal Parts in at least must be one of which the lemma, we e_1 derived equal Parts false. let be e_1, we e_1 derived equal Parts and Continue the Process. If this Process ends after a finite number of steps, we have arrived at a Contradiction and our lemma be true.

If However, the Process is Continuous, we get ~~sou~~ sequence of nested intervals $e_0, e_1, e_2 ---- e_n ---$ each Contained in the Preceding one for which the lemma is false. This sequence therefore determines a Point z_0 Common to all these squares such that z_0 is a limit Point of the set of Points in D. Since D is closed $z_0 \in D$.

As, f(z) diff = of z_0, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \;\forall z \quad \text{--- } (2)$$

Satisfying $|z - z_0| < \delta$, we can Choose ~~a~~ +ve integer N, Sufficiently large Such that the diagonal of the square $e_N < \delta$. Hence, for $n \geq N$, all the square e_n are Contained in ~~nhood neighood~~ nhd' $|z - z_0| < \delta$ also, $z_0 \in e_n$. Hence we arise the Contradiction since by taking z_1 to

$b \geq 0$ in the inequality ② is satisfied.
Hence, lemma is true.